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Spinning bound states of two and three black holes

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Abstract: Bound states of BPS particles in five-dimensional $\mathcal{N} = 2$ supergravity are counted by a topological index. We compute this bound state index exactly for two and three black holes as a function of the $\mathrm{SU}(2)_L$ angular momentum. The required regulator for the infrared continuum of near-coincident black holes is chosen in accord with the enhanced superconformal symmetry.

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Contents

1. Introduction

The study of semiclassical soliton scattering and moduli spaces has a long and rich history. A beautiful chapter, relevant to the present work, began with the realization that a pair of slowly-moving supersymmetric BPS monopoles is described by quantum mechanics on the two-monopole moduli space, which turns out to be the Atiyah-Hitchin space [1, 2]. The number of bound states is then determined by the moduli space cohomology, and is in agreement with predictions from S-duality [3].

It is natural to try to develop a similar picture for supersymmetric black holes. This problem is especially interesting because it provides a new angle to study the deep puzzles associated to quantum mechanical black holes. Work on construction of the N-black hole moduli space, which we shall denote $\mathcal{M}_N$, began in the early eighties [4, 5]. However the black hole problem turns out to be considerably more subtle than its monopole counterpart, in part because of divergences near the horizon at intermediate stages of the calculation. The supersymmetric moduli space for $N \geq 3$ has been found only very recently [6]–[9].[1]

Now that the moduli space is known, it is natural to try to compute the number of bound states of N black holes or, more reliably, the supersymmetric bound state index

$$\mathcal{I}^{(N)}(y) = \mathrm{Tr}\ y^{2J_L^3}(-)^{2J_R^3}, , \tag{1.1}$$

where the trace is over all states in the N black hole quantum mechanics and (J_L^3, J_R^3) are $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ angular momentum operators. Here one immediately encounters a puzzle. The moduli space quantum mechanics contains a divergent continuum of states describing highly redshifted, near-coincident black holes.[2] In order to compute $\mathcal{I}^{(N)}$ one must regulate this continuum. It is not obvious how the regulator should be chosen.

A relevant discovery in [6] is that this infrared continuum of states is in a representation of an enhanced superconformal symmetry. This observation is obviously pertinent to an understanding of the still-enigmatic AdS_2/CFT_1 correspondence, but the precise connection remains mysterious.[3] In [10] it was shown that the superconformal symmetry singles out a natural regulator for the infrared continuum. This regulator was then used to relate the index to 'superconformal cohomology' on the moduli space. Superconformal cohomology employs the nilpotent operator $\partial - D$, where the $(1,0)$ form D is associated to conformal scale transformations on $\mathcal{M}_N$. The cohomology was computed in [10] for the simplest case of two supersymmetric black holes in five dimensions with internal $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin $(0, 1/2)$.

In the present paper we present a solution of the superconformal bound state problem for two and three BPS black holes in five dimensions as a function of the spin eigenvalues (j_L, j_R). It is shown that for the N black hole problem the superconformal cohomology vanishes except at rank $N-1$. A cohomology generating function is then defined by

$$\mathcal{Z}^{(N)}(y,z) = \sum_{j_L, j_R} \mathrm{dim} H^{N-1}(\mathcal{M}_N, j_L, j_R) y^{2j_L} z^{2j_R}\,, \tag{1.2}$$

[1]The moduli space in [5] is inconsistent with supersymmetry at sixth order in the black hole masses and was corrected in [8].

[2]Near-horizon infrared divergences of this type have appeared in a variety of contexts in black hole physics. The presence of a divergent continuum of states, with a naively infinite capacity for information storage, is closely related to the information puzzle. Therefore we expect a proper understanding of how to regulate this infrared divergence to be relevant to the information puzzle.

[3]The enhanced superconformal symmetry group of the black hole quantum mechanics is the same as the superisometry group of $AdS_2 \times S^3$, indicating some relation of the former to the sought CFT_1 of AdS_2/CFT_1. However the CFT_1 envisioned in [16] was e.g. the quantum mechanics of a wrapped D-brane moduli space of the type considered in [15], which on the face of it is rather different.

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where the sum is over all the superconformal chiral primary states in the N-black hole quantum mechanics. We find

$$\mathcal{Z}^{(2)}(y,z) = \frac{yz}{(1-yz)(y-z)}$$
$$\mathcal{Z}^{(3)}(y,z) = 2\left[\frac{yz}{(1-yz)(y-z)}\right]^2 . \tag{1.3}$$

The generating function $\mathcal{Z}^{(N)}$ is not (as far as we know) invariant under supersymmetric corrections to the black hole quantum mechanics, so it is more natural to use the index $\mathcal{I}^{(N)}$ found by setting $z=-1$ in (1.3). This index is invariant under all deformations that preserve the superconformal structure of the quantum mechanics.

This paper is organized as follows. Section 2 contains a review of the geometry of the black hole moduli space, the superconformal structure, and the relation of the index to moduli space cohomology. This material is largely from [6, 10] and much of it is reviewed in [11]. In section 3 we show that the cohomology lives only in the middle dimension. In section 4 we calculate the bound state indexes $\mathcal{I}^{(2)}$ and $\mathcal{Z}^{(2)}$ for two black holes, generalizing the result of [10]. In section 5 we prove two more vanishing theorems that reduce the problem to the consideration of cohomology classes on certain subsets of the moduli space. In the three black hole case this cohomology may be found exactly using several Mayer-Vietoris type arguments, allowing us to compute $\mathcal{I}^{(3)}$ and $\mathcal{Z}^{(3)}$. In section 6 we raise the issue of uniqueness of our adopted definition of the index. We discuss an alternate definition which does not involve the superconformal structure and gives a trivial result (at least for $N=2$).

2. Review of superconformal black hole quantum mechanics

In this section we review pertinent results on the quantum mechanics of slowly moving black holes. We follow the notation of [10] where many of the statements are derived in more detail.

Consider N slowly moving BPS black holes in five-dimensional $\mathcal{N}=2$ supergravity with no matter. For sufficiently slow motion, the dynamics governing the relative positions is described by a quantum mechanical sigma model whose target space is the N-black hole moduli space.[4] Points on $\mathcal{M}_N$ are parameterized by the relative positions of the N black holes.[5] The quantum mechanics has four linearly realized

[4]For simplicity we here and hereafter ignore the center-of-mass degrees of freedom.

[5]In the N-monopole problem, the moduli space has an asymptotic identification under the permutation group S_N, corresponding to the fact that the monopoles are identical particles. This identification is required for smoothness of the moduli space in the interior. In the black hole case, in contrast, the moduli space is smooth without identifications. Implementing S_N identifications could induce extra cohomology above and beyond what we find herein. Whether or not this is appropriate may depend on microscopic considerations and cannot be semiclassically determined [12]. In this paper we do not consider such identifications.

Poincare supersymmetries inherited from the four spacetime symmetries which are unbroken by the BPS black holes. It also has an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ global symmetry arising from spatial rotations and an additional R-symmetry which we denote $\mathrm{SU}(2)_I$.[6]

At very low energies one finds that the theory splits into two different types of decoupled sectors. One describes noninteracting, freely moving black holes while the other describes strongly interacting, near-coincident black holes. The near-coincident quantum mechanics has an enhanced superconformal symmetry $D(2,1;0)$ which has eight supercharges and incorporates $\mathrm{SU}(2)_R$ spatial rotations [13, 6]. We denote the near-coincident moduli space $\mathcal{M}_N$. The superconformal structure highly constrains the geometry of $\mathcal{M}_N$ [14] as we now describe.

2.1 The geometry of $\mathcal{M}_N$

The near-coincident N-black hole moduli space $\mathcal{M}_N$ has a triplet of self-dual complex structures obeying

$$I^r I^s = -\delta^{rs} + \epsilon^{rst} I^t \,, \tag{2.1}$$

for $r, s = 1, 2, 3$. The complex dimension is given by

$$\dim_{\mathbb{C}}(\mathcal{M}_N) = 2N - 2 = n \,. \tag{2.2}$$

The metric is

$$g_{a\bar{b}} = \frac{1}{2}(\partial_a \partial_{\bar{b}} L + I_a^{-\bar{c}} I_{\bar{b}}^{+d} \partial_{\bar{c}} \partial_d L) \,. \tag{2.3}$$

In this expression, the complex coordinates

$$z^a, z^{\bar{a}} \,, \qquad\qquad a = 1, 2, \ldots, n \tag{2.4}$$

are adapted to I^3, and $I^{\pm} = (1/2)(I^1 \pm iI^2)$. The z^a are built out of the real coordinates of the black holes $\vec{X}^A, A = 1, \ldots, N$ after factoring out the center of mass in the usual way. We will use indices $a, b, \ldots$ for the n complex coordinates and $M, N, \ldots$ for the $2n$ real coordinates. L is a function of the black hole positions given by[7]

$$L = -\int d^4X \left(\sum_{A=1}^{N} \frac{Q}{|\vec{X} - \vec{X}^A|^2} \right)^3 \,. \tag{2.5}$$

Although this function is at first sight divergent, the infinite part of L does not contribute to the metric (2.3) — we refer the reader to appendix A for details. Removing this irrelevant part, we find that L obeys

$$(z^a \partial_a + 1) L = -\frac{1}{2} K \,, \tag{2.6}$$

[6] In [10] this R-symmetry was denoted $\mathrm{SU}(2)_R$.

[7] For simplicity we have set all the black hole charges equal to Q.

where K is the function

$$K = 6\pi^2 \sum_{A \neq B}^{N} \frac{Q^3}{|\vec{X}^A - \vec{X}^B|^2} \,. \tag{2.7}$$

The metric (2.3) has a complex homothety generated by the Lie derivative $\mathcal{L}_D g_{a\bar{b}} = g_{a\bar{b}}$, which acts on scalars as

$$\mathcal{L}_D \sim -z^a \partial_a \,. \tag{2.8}$$

We use D^a ($D^{\bar{a}}$) to denote the homothetic vector field $-z^a$ ($-z^{\bar{a}}$) and $\bar{D}$ (D) to denote the associated $(0,1)$ ($(1,0)$) form constructed with the metric $g_{a\bar{b}}$. The norm of D is just

$$D_a D^a = K \,. \tag{2.9}$$

The imaginary part of (2.8) is part of an $\mathrm{SU}(2)_R$ triplet of isometries generated by

$$\mathcal{L}_{D^r} \sim X^M I^{rN}_M \partial_N \,, \qquad r = 1, 2, 3 \,. \tag{2.10}$$

There are also $\mathrm{SU}(2)_L$ isometries generated by

$$X^M K^{rN}_M \partial_N \,, \qquad\qquad r = 1, 2, 3 \,, \tag{2.11}$$

where K^r are the triplet of anti-self dual complex structures.

Using the metric and the complex structures, one may construct holomorphic and antiholomorphic two-forms

$$I^- = \frac{1}{2} I^-_{ab} dz^a dz^b \,, \qquad\qquad I^+ = \frac{1}{2} I^+_{\bar{a}\bar{b}} dz^{\bar{a}} dz^{\bar{b}} \,. \tag{2.12}$$

I^- obeys the relations

$$\partial I^- = 0 \,, \qquad \partial * e^{-\phi} I^- = 0 \,, \tag{2.13}$$

with $*$ the Hodge dual.[8] In the $\vec{X}^A$ coordinates ϕ is given by

$$e^{2\phi} = \sqrt{\det g} \,. \tag{2.14}$$

2.2 The Hilbert space as $(p, 0)$-forms

The Hilbert space of the black hole quantum mechanics can be identified with the space of $(p, 0)$-forms f_p on $\mathcal{M}_N$. The inner product is

$$\langle f'_p | f_p \rangle = \int_{\mathcal{M}_N} e^{-\phi - 2K} f'_p \wedge * f_p \,. \tag{2.15}$$

[8] In contrast to the conventions of [10], we here include complex conjugation.

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The action of the superconformal algebra is simply represented on $(p,0)$-forms. $D(2,1;0)$ is the semidirect product of $\mathrm{SU}(1,1|2)$ and $\mathrm{SU}(2)_I$. The $\mathrm{SU}(2)_I$ is generated by the Lefschetz action of $(i/2)I^-$,

$$
\begin{aligned}
\frac{i}{2}I^3 f_p &= \frac{n-2p}{4} f_p \,, \\
\frac{i}{2}I^- f_p &= \frac{i}{2}I^- \wedge f_p \,, \\
\frac{i}{2}I^+ f_p &= -\frac{i}{2} * I^- \wedge * f_p \,,
\end{aligned}
\tag{2.16}
$$

The bosonic R-symmetry in $\mathrm{SU}(1,1|2)$ is $\mathrm{SU}(2)_R$, and is generated by the Lie derivatives (2.10). The supercharges are in the $(2,2)$ of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$. The operator L_0 is the hamiltonian associated to the metric (2.3) with potential (2.9). The actions of the weight $(1/2,-1/2)$ charge and its adjoint with respect to (2.15) are

$$
\begin{aligned}
G^{+-}_{-1/2} f_p &= -\frac{i}{\sqrt{2}}(\partial f_p - 2D \wedge f_p) \,, \\
G^{-+}_{1/2} f_p &= \frac{i}{\sqrt{2}} \partial^\dagger f_p \,,
\end{aligned}
\tag{2.17}
$$

where

$$
\partial^\dagger \equiv - * e^{\phi} \partial e^{-\phi} * \,. \tag{2.18}
$$

Commutators of these basic operators then generate the full algebra.

2.3 Chiral primaries

A chiral primary is a normalizable $(p,0)$-form that is annihilated by both $G^{+-}_{-1/2}$ and its adjoint $G^{-+}_{1/2}$. From (2.17) this is equivalent to the harmonic condition

$$
\partial f_p - 2D \wedge f_p = \partial^\dagger f_p = 0 \,. \tag{2.19}
$$

Thus the chiral primaries are harmonic representatives of $H^p(\mathcal{M}^N)$, which we define to be the cohomology of normalizable $(p,0)$-forms on $\mathcal{M}^N$ relative to the differential operator $\partial - 2D$. The $D(2,1;0)$ algebra can be used to show that a normalizable solution of (2.19) is annihilated by J^+_R, $L_0 - J^3_R$, L_1 and all the supercharges except for $G^{-\pm}_{-1/2}$. This implies the relations

$$
\partial f_p = 0 \,, \tag{2.20}
$$

$$
D \wedge f_p = 0 \,, \tag{2.21}
$$

$$
(D^a \partial_a - D^{\bar{a}} \partial_{\bar{a}}) f_p = 2 j_R f_p \,, \tag{2.22}
$$

$$
D^{\bar{a}} I^{+b}_{\bar{a}} \partial_b f_p = 0 \,, \tag{2.23}
$$

where j_R is the eigenvalue of and f_p is an eigenform of J^3_R. These last two equations follow from our choice of complex coordinates adapted to J^3_R in which $D^a = -z^a$. Further useful relations may be found in [10]. A chiral primary with $j_R = 0$ would be an $\mathrm{SU}(2)_R$ singlet annihilated by all eight supercharges. It is easy to see that this is impossible. Hence j_R is strictly positive.

2.4 The bound state index

For theories with eight supercharges in five dimensions, the weighted numbers of BPS states are given by an index — roughly the difference in the numbers of hypermultiplets and vector multiplets. This index is invariant under smooth deformations of the theory. It has been computed in some examples of M-theory compactifications in [15]. For the case of multi-black hole bound states, a prescription must be given for regulating the infrared continuum of near-coincident black holes. In [10] a regulator was proposed which amounts to working in a basis of L_0 rather than hamiltonian eigenstates. L_0 differs from the hamiltonian by the potential K (equation (2.7)) which eliminates the infrared divergences. With this prescription computing the index reduces to counting the chiral primaries, weighted by $(-)^{2j_R}$. Since the $\mathrm{SU}(2)_L$ generators J_L commute with all the superconformal generators, the index can be refined by inclusion of a J_L^3 weighting factor

$$\mathcal{I}^{(N)}(y) = \mathrm{Tr}(-)^{2J_R^3} y^{2J_L^3} \,. \tag{2.24}$$

The quantum mechanics on $\mathcal{M}_N$ and hence expression (2.24) does not include the center of mass degrees of freedom of the N black holes. Including this would multiply (2.24) by a factor of $(y^{1/2} + y^{-1/2})^2$.

3. A vanishing theorem

In this section, we show that chiral primaries f_p exist only at middle rank. That is, $p = n/2 = N - 1$. We will take f_p to be a J_R^3 eigenform with eigenvalue j_R and hence obeying (2.22). Our strategy will be to first show that all chiral primaries are annihilated by I^-.

Given a chiral primary f_p, it is straightforward to verify that the action of I^-

$$I^- \wedge f_p \tag{3.1}$$

generates a new chiral primary (if it does not vanish). Using $I^- = -\partial\partial^- L$, with $\partial^- \sim dz^a I_a^{-\bar{b}} \partial_{\bar{b}}$ and L the finite expression in appendix A, the norm is

$$\langle I^- \wedge f_p | I^- \wedge f_p \rangle = \int e^{-\phi - 2K} (\partial\partial^- L \wedge f_p) \wedge *\partial\partial^- L \wedge f_p \,. \tag{3.2}$$

If boundary terms can be ignored, this integral vanishes upon integration by parts with respect to ∂. Note that the factor of e^{-2K} in the integral insures that the boundary terms vanish at large K.

We wish to show that the boundary terms vanish at small K as well. To do this, we will introduce cutoff functions into the integral and then show that the error terms they introduce can be made arbitrarily small. Let $\rho_\mu : (0, \infty) \to [0, 1]$ be a sequence of differentiable compactly supported cutoff functions such that $\rho_\mu(t) = 1$ for $t \in [1/\mu, \mu]$ and such that

$$|\rho'_\mu(t)| < \frac{1}{t|\ln t|}\,. \tag{3.3}$$

With these hypotheses, we have

$$\langle I^- \wedge f_p | I^- \wedge f_p\rangle = \lim_{\mu\to\infty} \langle \partial\partial^- L \wedge f_p | \rho_\mu(K) I^- \wedge f_p\rangle\,. \tag{3.4}$$

The cutoff function allows us to integrate by parts obtaining

$$\lim_{\mu\to\infty} \langle \partial^- L \wedge f_p | [\partial^\dagger, \rho_\mu(K)] I^- \wedge f_p\rangle\,. \tag{3.5}$$

The derivative of the cutoff function is converging to zero pointwise. Hence, (given the finiteness of $||f_p||^2$) the above limit vanishes if we can show that $|\partial^- L||[\partial^\dagger, \rho_\mu(K)]|$ is bounded independent of μ.

First we note using (2.9) that

$$|[\partial^\dagger, \rho_\mu(K)]| = |\rho'_\mu(K)\partial K| < \frac{1}{K^{1/2}|\ln K|}\,. \tag{3.6}$$

On the other hand, we show in appendix A that

$$\frac{|\partial^- L|}{K^{1/2}|\ln K|} < c_N\,, \tag{3.7}$$

for some constant c_N depending only on the number of black holes.

Thus, the integral (3.2) vanishes. Since the integrand is nonnegative it must vanish pointwise. Consistency then demands that all chiral primaries obey

$$I^- \wedge f_p = 0\,. \tag{3.8}$$

Comparing with (2.16) we see that this is equivalent to the statement that f_p is a lowest weight $\mathrm{SU}(2)_I$ state annihilated by I^-. On the other hand, since $\mathrm{SU}(2)_I$ mixes the supercharges which annihilate a chiral primary only among themselves, chiral primaries are representations of $\mathrm{SU}(2)_I$. It then follows from (3.8) that chiral primaries must be I-singlets, and hence have

$$p = \frac{n}{2}\,. \tag{3.9}$$

This can be expressed as the vanishing theorem for superconformal cohomology

$$H^p(\mathcal{M}_N) = 0\,, \qquad p \neq N-1\,. \tag{3.10}$$

4. The index for two black holes

For $N = 2$ we can find the chiral primaries (i.e. the group $H^1(\mathcal{M}_2)$) by direct computation. The metric is

$$ds^2 = 2c\frac{d\vec{X}\cdot d\vec{X}}{|\vec{X}|^4}\,, \tag{4.1}$$

where $\vec{X} \equiv \vec{X}^1 - \vec{X}^2$ is the relative separation of the black holes, and $c = 12\pi^2 Q^3$ is a constant. We have

$$\vec{D} = -\vec{X}\,, \qquad K = c|\vec{X}|^{-2}\,, \qquad e^{-\phi} = c^{-1}|\vec{X}|^4\,. \tag{4.2}$$

We denote the usual complex coordinates built out of the $\vec{X}$ as z^1 and z^2.

For $n = 2$, we find that the chiral primary conditions (2.20)–(2.23) are satisfied only if $f = \alpha K^{2j_R-1}D$ for some function α obeying

$$\partial\alpha = 0\,, \qquad D^{\bar{a}}\partial_{\bar{a}}\alpha = (1-2j_R)\alpha\,. \tag{4.3}$$

We can calculate the norm (2.15)

$$\begin{aligned}\langle f|f\rangle &= \int d^4X\sqrt{g}e^{-\phi}|\alpha K^{2j_R-1}|^2 K e^{-2K}\\ &= c^{4j_R}\int d^4X|\alpha|^2|\vec{X}|^{-2-8j_R}e^{-2c|\vec{X}|^{-2}}\\ &= c^{4j_R}\int dr r^{-1-4j_R}e^{-2cr^{-2}}\int_{S^3}d^3\Omega|\hat{\alpha}|^2\\ &= \frac{1}{2}\left(\frac{c}{2}\right)^{2j_R}\Gamma(2j_R)\int_{S^3}d^3\Omega|\hat{\alpha}|^2\end{aligned} \tag{4.4}$$

where $\hat{\alpha}$ denotes α restricted to the unit 3-sphere $S^3 = \{|\vec{X}| = 1\}$. Thus $j_R \geq 1/2$ and $\hat{\alpha}$ must be integrable on S^3. Clearly α must be a homogeneous polynomial in $\bar{z}^1$ and $\bar{z}^2$ of order $2j_R - 1$, since singularities of the form $(\bar{z}^1)^{-1}$ and $(\bar{z}^2)^{-1}$ are irregular and would cause the norm to diverge. There is a basis of $2j_R$ such polynomials: $\alpha = (\bar{z}^1)^{2j_R-1}, (\bar{z}^1)^{2j_R-2}(\bar{z}^2), \ldots, (\bar{z}^2)^{2j_R-1}$. We may choose our basis of J_L generators so that these polynomials have J^3_L eigenvalues $j_R - 1/2, j_R - 3/2, \ldots, -j_R + 1/2$, respectively. So at level j_R there are $2j_R$ chiral primaries that form an irreducible $\mathrm{SU}(2)_L$ multiplet of maximal spin $j_L = j_R - 1/2$. This is summarized by

$$\begin{aligned}\dim H^1(\mathcal{M}_2, j_L, j_R) &= 1\,, \qquad j_R \geq \frac{1}{2}\,, \qquad |j_L| < j_R\,, \qquad \text{and } j_L + j_R + \frac{1}{2} \in \mathbb{Z}\\ &= 0 \qquad \text{otherwise}\,.\end{aligned} \tag{4.5}$$

We adopt here the notation that $H^p(\mathcal{M}_N, j_L, j_R)$ is the restriction of the cohomology to J^3_L and J^3_R eigenspaces. This reproduces the result of [10] for $j_R = 1/2$. Summing

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the chiral primaries with fixed j_R weighted by $y^{2J_L^3}$ gives

$$(-)^{2j_R}\sum_{k=-j_R+1/2}^{k=j_R-1/2} y^{2k} = (-)^{2j_R}\frac{y^{2j_R}-y^{-2j_R}}{y-y^{-1}}\,. \tag{4.6}$$

The index (2.24) for $N=2$ is then obtained by summing over j_R:

$$\mathcal{I}^{(2)}(y) = \mathrm{Tr}(-)^{2J_R^3}y^{2J_L^3} = -\frac{1}{(y^{1/2}+y^{-1/2})^2}\,. \tag{4.7}$$

The index (4.7) does not generate all of the cohomology because of the unweighted sum over j_R. A generating partition function for all the cohomology can be defined by

$$\mathcal{Z}^{(2)}(y,z) = \sum_{j_L,j_R} \mathrm{dim}H^1(\mathcal{M}_2, j_L, j_R)z^{2j_R}y^{2j_L}\,. \tag{4.8}$$

This partition function is not in general a supersymmetric invariant index of the black hole quantum mechanics, but nevertheless usefully summarizes the results of our computation. For $N=2$ we have

$$\mathcal{Z}^{(2)}(y,z) = \frac{yz}{(1-yz)(y-z)}\,. \tag{4.9}$$

5. The index for three black holes

The computation of the index for $N>2$ black holes is considerably more involved. We first prove two more vanishing theorems that hold in the general N case. We then apply these to the case of three black holes and, using several Mayer-Vietoris type arguments, compute the bound state index.

5.1 Two more vanishing theorems

In this subsection we consider two appropriately defined regions V_N and W_N of $\mathcal{M}_N$ and find that the Neumann cohomology on these subsets is trivial for positive j_R. Here, V_N is the region near the singularities of the function K (i.e. near-coincident black holes) and W_N is the region where K is small (i.e. widely separated black holes). This result will enter into the exact sequence for the cohomology derived in the next subsection.

It is convenient to work in terms of rescaled forms

$$h_p = e^{-K}f_p\,, \tag{5.1}$$

which we take to obey Neumann boundary conditions on the region V_N, so that the pullback of $*h_p$ to the boundary of V_N vanishes.[9] The inner product (2.15) reduces

[9] This condition on $*h_p$ follows from the requirement that h_p be in the domain of $\partial^\dagger$, i.e. for all $p-1$ forms g on V_N we have $\langle\partial g, h\rangle < c|g|$ for some (g independent) c.

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to

$$\langle h'|h\rangle_{V_N} = \int_{V_N} e^{-\phi} h'_p \wedge * h_p \,. \tag{5.2}$$

If f_p is a cohomology element relative to $\partial - 2D$ then h_p is a cohomology element relative to $\partial - D$, i.e.

$$(\partial - D) h_p = 0 \,. \tag{5.3}$$

We consider the functional $E(h_p)$ defined as

$$\begin{aligned} E &= ||(\partial - D)h_p||^2_{V_N} + ||(\partial^\dagger - i_D)h_p||^2_{V_N} \\ &= ||\partial h_p||^2_{V_N} + ||\partial^\dagger h_p||^2_{V_N} + \langle h_p\{D, i_D\}|h_p\rangle_{V_N} - 2\mathrm{Re}\langle h_p\{\partial, i_D\}|h_p\rangle_{V_N} \,, \end{aligned} \tag{5.4}$$

where $i_D = *D*$ is the adjoint of the wedge product with $D = \partial K$. In this expression the norm is determined from (5.2). In the second line a boundary term which vanishes due to the Neumann condition has been dropped in integrating by parts. Using $\{\partial, i_D\} = \mathcal{L}_D$, $\nabla_a e^{-\phi} D^a = 0$, and integrating by parts the last term in (5.4) can be written

$$-2\mathrm{Re}\langle h_p|\mathcal{L}_D h_p\rangle_{V_N} = -2j_R ||h_p||^2_{V_N} - \mathrm{Re}\int_{\partial V_N} *De^{-\phi}|h_p|^2 \,. \tag{5.5}$$

In writing (5.5) we assume the boundary ∂V_N is invariant under the action of J^3_R so that h_p can be taken to be a J^3_R eigenform with eigenvalue j_R. Using $\{D, i_D\} = K$ together with (5.5) in (5.4) yields

$$E = ||\partial h_p||^2_{V_N} + ||\partial^\dagger h_p||^2_{V_N} + ||\sqrt{K} h_p||^2_{V_N} - 2j_R||h_p||^2_{V_N} - \mathrm{Re}\int_{\partial V_N} *De^{-\phi}|h_p|^2 \,. \tag{5.6}$$

So far we have said nothing about the region V_N. If V_N satisfies[10]

$$V_N \subset \{K > 2j_R + 1\} \,, \tag{5.7}$$

so that V_N is near the region where one or more black holes are coincident, then the sum of third and fourth terms in (5.6) will be greater than $||h_p||^2$. Also, note that the last term in (5.6) is nonnegative if the outward unit normal to V_N, call it n^V, obeys

$$D^a n^V_a < 0 \,. \tag{5.8}$$

So in this case

$$E(h_p) > ||h_p||^2_{V_N} \,. \tag{5.9}$$

It is a theorem from complex analysis that the bound (5.9) implies the vanishing of cohomology (see, e.g. [17, section 4.4]). We therefore conclude that if V_N obeys (5.7) and (5.8), then the Neumann cohomology of V_N is trivial,

$$H^p(V_N) = 0 \,. \tag{5.10}$$

[10] This condition on V_N of course has a j_R dependence, but we suppress this in the following to avoid cluttering the equations.

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The simplest example of such a region is just $V_N = \{K > 2j_R + 1\}$.

We now consider the opposite case, namely a region W_N such that

$$W_N \subset \{K < a\}\,, \tag{5.11}$$

for some constant a. In this case, we may conjugate the differential operator $\partial - D$ by e^K to get $e^{-K}(\partial - D)e^K = \partial$, the usual differential operator. The function e^K is bounded on W_N, so the cohomology is unchanged.[11] We therefore need to show that the functional

$$E = ||\partial h_p||^2_{W_N} + ||\partial^\dagger h_p||^2_{W_N} \tag{5.12}$$

satisfies the appropriate bound. First, note that for any number α we may follow the logic of (5.4) to get

$$\begin{aligned} E = {} & ||(\partial - \alpha D)h_p||^2_{W_N} + ||(\partial^\dagger - \alpha i_D)h_p||^2_{W_N} + \\ & +2\alpha \mathrm{Re}\langle h_p\{\partial, i_D\}|h_p\rangle_{W_N} - \alpha^2\langle h_p\{D, i_D\}|h_p\rangle_{W_N} \end{aligned} \tag{5.13}$$

so that by (5.11)

$$E \geq 2\alpha \mathrm{Re}\langle h_p\{\partial, i_D\}|h_p\rangle_{W_N} - \alpha^2 a||h_p||^2_{W_N}\,. \tag{5.14}$$

Let us further demand that

$$D^a n^W_a > 0\,, \tag{5.15}$$

where n^W is the outward directed normal to W_N. The integral $\int_{\partial W_N} *De^{-\phi}|h_p|^2$ is positive, so (5.5) implies that

$$Re\langle h_p\{\partial, i_D\}|h_p\rangle \geq j_R||h_p||^2_{W_N} \tag{5.16}$$

for any j_R and h_p obeying Neumann boundary conditions. Equation (5.14) then becomes

$$E \geq (2\alpha j_R - \alpha^2 a)||h_p||^2_{W_N}\,. \tag{5.17}$$

For any $j_R > 0$ we may choose an α small enough that $(2\alpha j_R - \alpha^2 a) > 0$, so (5.17) implies the vanishing of cohomology. We thus conclude that if the region W_N satisfies (5.11) and (5.15), then the positive charge Neumann cohomology of W_N is trivial,

$$H^p(W_N) = 0 \qquad \text{for } j_R > 0\,. \tag{5.18}$$

The simplest example of such a region is just $W_N = \{K < a\}$ for some constant a.

[11] This deformation of the differential operator may be equivalently viewed as multiplication of the metric by some function of K, which is bounded on W_N. This new metric is quasiisometric to the old, so the cohomology is unchanged.

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5.2 Exact sequences relating subsets of $\mathcal{M}_N$

We will now use these two vanishing theorems to study the cohomology of $\mathcal{M}_N$. Consider a W_N satisfying (5.11) and (5.15). Hodge duality sends $p \to n-p$, $(j_L, j_R) \to (-j_L, -j_R)$ and interchanges Neumann and Dirichlet boundary conditions, so[12]

$$H^p(W_N, j_L, j_R) = H_D^{n-p}(W_N, -j_L, -j_R)\,. \tag{5.19}$$

The exact sequence of forms

$$0 \longrightarrow \Omega_D(W_N, j_L, j_R) \longrightarrow \Omega(\mathcal{M}_N, j_L, j_R) \longrightarrow \Omega(\mathcal{M}_N \setminus W_N, j_L, j_R) \longrightarrow 0 \tag{5.20}$$

induces a long exact sequence relating the cohomology of W_N to that of its complement $\mathcal{M}_N \setminus W_N$. However, if we choose W_N such that $\mathcal{M}_N \setminus W_N$ satisfies (5.7) and (5.8) then our vanishing theorem on $\mathcal{M}_N \setminus W_N$ assures us that

$$\begin{aligned} H^p(\mathcal{M}_N, j_L, j_R) &= H_D^p(W_N, j_L, j_R) \\ &= H^{n-p}(W_N, -j_L, -j_R)\,. \end{aligned} \tag{5.21}$$

For future reference, let us apply this formula in the two black hole case, with $W_2 = \{\vec{x} : |\vec{x}| > c\}$. Plugging in the result (4.5) we find that W_2 has nonzero cohomology

$$H^1(W_2, j_L, j_R) = C\,, \qquad \text{for } j_R < 0\,, \quad |j_L| < |j_R|\,, \qquad \text{and } j_L + j_R + \frac{1}{2} \in \mathbb{Z}\,. \tag{5.22}$$

We will also make use of a slightly modified construction. For a choice of V_N and W_N that satisfies (5.7), (5.8), (5.11) and (5.15), and whose union is the entire space $\mathcal{M}_N$, define the region Y_N to be

$$Y_N = V_N \cap W_N\,. \tag{5.23}$$

Y_N resembles a shell surrounding the singularities of K. We have the short exact sequence of complexes

$$0 \longrightarrow \Omega(\mathcal{M}_N) \xrightarrow{r} \Omega(W_N) \oplus \Omega(V_N) \xrightarrow{s} \Omega(Y_N) \longrightarrow 0\,, \tag{5.24}$$

where r denotes the restriction map and s denotes the subtraction map. Both r and s are compatible with the differential operator $\partial - D$, so the usual arguments give the Mayer-Vietoris sequence for $\partial - D$ cohomology,

$$\cdots \longrightarrow H^p(\mathcal{M}_N) \longrightarrow H^p(V_N) \oplus H^p(W_N) \longrightarrow H^p(Y_N) \longrightarrow H^{p+1}(\mathcal{M}_N) \longrightarrow \cdots\,. \tag{5.25}$$

[12] One must take care when applying Hodge duality to cohomology with respect to the operator $\partial - D$. If a harmonic form h is annihilated by $\partial - D$ and $\partial^\dagger - i_D$ then $*h$ is annihilated by $\partial + D$ and $\partial^\dagger + i_D$. So in general Hodge duality will not interchange cohomology classes. However, on regions where K is bounded then we may multiply forms by a factor of e^K and reduce to usual ∂-cohomology. Thus Hodge duality allows us to relate Dirichlet and Neumann cohomology on W_N, but not on V_N or $\mathcal{M}_N$.

Plugging in (5.10) and (5.18) gives

$$H^{N-1}(\mathcal{M}_N) = H^{N-2}(Y_N) \qquad \text{for } j_R > 0\,. \tag{5.26}$$

We saw in section 2.3 that all chiral primaries have $j_R > 0$, so (5.26) gives the complete cohomology of $\mathcal{M}_N$.

5.3 Exact Sequence for $\mathcal{M}_3$ Cohomology

Let us apply the result (5.26) of the last subsection to compute the cohomology of $\mathcal{M}_3$. Define $W_3^c = \{\mathbf{\vec{x}} : \ |\vec{x}^1| > c,\ |\vec{x}^2| > c,\ |\vec{x}^1 - \vec{x}^2| > 4c\}$ and $V_3^c = \{\mathbf{\vec{x}} : \ |\vec{x}^1| < 2c\} \cup \{\mathbf{\vec{x}} : \ |\vec{x}^2| < 2c\} \cup \{\mathbf{\vec{x}} : \ |\vec{x}^1 - \vec{x}^2| < 8c\}$. For any value of j_R we may choose a value of c such that these regions satisfy $V_3^c \cup W_3^c = \mathcal{M}_3$ as well as the conditions (5.7), (5.8), (5.11), (5.15).

We identify Y_3 as the union of three subspaces U_1, U_2, U_3, and use Mayer-Vietoris sequences to compute its cohomology. The three subspaces are defined as

$$\begin{aligned} U_1 &= W_3^c \cap \{\mathbf{\vec{x}} : \ |\vec{x}^1| < 2c\} \\ U_2 &= W_3^c \cap \{\mathbf{\vec{x}} : \ |\vec{x}^2| < 2c\} \\ U_3 &= W_3^c \cap \{\mathbf{\vec{x}} : \ |\vec{x}^1 - \vec{x}^2| < 8c\}\,. \end{aligned} \tag{5.27}$$

Geometrically, the region U_1 ($\cong U_2$) looks like a thickened cylinder with a single hole removed, whereas U_3 looks like a thickened cylinder with two holes removed. Notice that $U_1 \cap U_2 = \emptyset$. Therefore $H^p(U_1 \cup U_2) = H^p(U_1) \oplus H^p(U_2)$. The long exact sequence derived from $Y_3 = (U_1 \cup U_2) \cup U_3$, which is

$$\cdots \to H^p(Y_3) \to H^p(U_1 \cup U_2) \oplus H^p(U_3) \to H^p((U_1 \cup U_2) \cap U_3) \to H^{p+1}(Y_3) \to \cdots \tag{5.28}$$

reduces to

$$\begin{aligned} \cdots \to H^p(Y_3) \to H^p(U_1) \oplus H^p(U_2) \oplus H^p(U_3) \to H^p(U_1 \cap U_3) \oplus H^p(U_2 \cap U_3) \to \\ \to H^{p+1}(Y_3) \to \cdots\,. \end{aligned} \tag{5.29}$$

So we must compute the cohomology of the three spaces U_1 ($\cong U_2$), U_3, and $U_1 \cap U_3$ ($\cong U_2 \cap U_3$). One more definition will help. For $i = 1, 2$, let

$$U_3^i = \{\mathbf{\vec{x}} : \ 4c < |\vec{x}^1 - \vec{x}^2| < 8c,\ |\vec{x}^i| > c\}\,. \tag{5.30}$$

This includes a region that is singular with respect to the usual metric on $\mathcal{M}_3$; for example, $U_3^1 = U_3 \cup \{\mathbf{\vec{x}} : \ 4c < |\vec{x}^1 - \vec{x}^2| < 8c,\ |\vec{x}^1| > c,\ |\vec{x}^2| \le c\}$ contains the $\vec{x}^2 = 0$ singularity. We will remedy this by taking the metric on the $|\vec{x}^2| \le c$ component to be non-singular; we replace the $\frac{(d\vec{x}^2)^2}{|\vec{x}^2|^4}$ term in the metric by $(d\vec{x}^2)^2$. Roughly speaking, U_3^1 is a cylindrical tube with a hole removed at $|\vec{x}^1| \le c$. With the change of variables $\vec{x}^1 \to \vec{x}^1 - \vec{x}^2$, $\vec{x}^2 \to \vec{x}^1$ this region looks like U_2, and in fact with the non-singular metric on U_3^1 these two regions are quasi-isometric. We conclude that for the purposes of cohomology, $U_3^i \cong U_1 \cong U_2$. Moreover, note that $U_3^1 \cap U_3^2 = U_3$ with the correct $\mathcal{M}_3$ metric, so

$$\cdots \to H^{p-1}(U_3) \to H^p(U_3^1 \cup U_3^2) \to H^p(U_3^1) \oplus H^p(U_3^2) \to H^p(U_3) \to \cdots . \quad (5.31)$$

First, let us compute the U_3^i cohomology. U_3^i is the product of B^* (a punctured four ball) and $\{\vec{\mathbf{x}} : \ |x^i| > c\}$. This base is W_2, so the Künneth formula gives

$$\begin{aligned} H^p(U_3^i) &= \bigoplus_q H^q(B^*) \otimes H^{p-q}(W_2) \\ &\simeq H^0(B^*) \otimes H^p(W_2) , \end{aligned} \quad (5.32)$$

where we use the fact that W_2 has cohomology only at negative j_R, so that only the positive charge cohomology of B^* will contribute (see appendix B) if we evaluate $H^p(U_3^i)$ at nonnegative j_R. We conclude from (5.22) that the only nonvanishing cohomology is at $p = 1$. But note further that we can replace $H^1(W_2)$ by $H^1(B^*)$. Therefore the quantities $H^p(U_1) \oplus H^p(U_2)$ in (5.29) and $H^p(U_3^1) \oplus H^p(U_3^2)$ in (5.31) can both be replaced by $H^1(B^* \times B^*)$ for $p = 1$ and 0 for $p \neq 1$.

Second, we will use (5.31) to compute $H^p(U_3)$. Let us see that $H^p(U_3^1 \cup U_3^2) = 0$. This space is a B^* fibration over the space $S = \{\vec{\mathbf{x}} : \ \vec{x}^1 = \vec{x}^2\}$. We will show that $H^p(S) = 0$. For computing cohomology, the space S can be identified with $B \cup W_2$. The intersection is $B \cup W_2 \simeq B^*$. From the preceding subsection, we know that cohomology of W_2 vanishes for non-negative j_R. Hence for charge $j_R \geq 0$ the relevant exact sequence reduces to

$$\begin{aligned} 0 &\to H^0(S) \to H^0(B) \to H^0(B^*) \to H^1(S) \to H^1(B) \to H^1(B^*) \\ &\to H^2(S) \to H^2(B) \to H^2(B^*) \to 0 . \end{aligned} \quad (5.33)$$

From the results of appendix B it follows that $H^2(S) = H^1(B^*)$, which is zero for $j_R \geq 0$. Moreover, the map $H^0(B) \to H^0(B^*)$ is simply the restriction map. Hartog's theorem states that in complex dimension greater than 1, holomorphic functions on a domain D minus an interior ball extend across the entire domain D, hence this restriction map is an isomorphism. This implies that $H^0(S) = H^1(S) = 0$, so $H^p(U_3^1 \cup U_3^2) = 0$. At negative j_R we apply Hodge duality to the positive j_R results, and conclude that $H^p(S) = 0$ for all j_R. Substituting the results of the previous two paragraphs into (5.31), we find that the only nonvanishing cohomology of U_3 is

$$H^1(U_3) = H^1(B^* \times B^*) . \quad (5.34)$$

Third, we notice that

$$H^p(U_1 \cap U_3) = H^p(B^* \times B^*) . \quad (5.35)$$

This is clear because $U_1 \cap U_3$ is defined by the conditions $c < |\vec{x}^1| < 2c$ and $4c < |\vec{x}^1 - \vec{x}^2| < 8c$; these already imply the third condition, $|\vec{x}^2| > c$. So $U_1 \cap U_3 \cong U_2 \cap U_3$ is biholomorphic to $B^* \times B^*$.

Now we are ready to substitute all these results into (5.29). Recall from the preceding section that the only nonvanishing cohomology of Y_3 is at $p = 1$. The long exact sequence reduces to

$$0 \to 2H^0(B^* \times B^*) \to H^1(Y_3) \to 2H^1(B^* \times B^*) \to 2H^1(B^* \times B^*) \to 0\,. \tag{5.36}$$

Therefore $H^1(Y_3) \simeq 2H^0(B^* \times B^*)$. Using (5.26), we conclude that

$$H^2(\mathcal{M}_3) \simeq 2H^0(B^* \times B^*)\,. \tag{5.37}$$

It is useful to further refine the equation (5.37). The cohomology appearing in any exact sequence can be restricted to eigenspaces of an operator which commutes with $\partial - D$. In the case at hand, for p-th cohomology, two such operators are J_L^3 and $J_R^3 - p/2$, whose eigenvalues are denoted j_L and $j_R - p/2$. We then have the relation

$$H^2(\mathcal{M}_3, j_L, j_R) = 2H^0(B^* \times B^*, j_L, j_R - 1)\,. \tag{5.38}$$

In this expression the second and third arguments of the cohomology groups are the J_L^3 and J_R^3 eigenvalues, respectively.

5.4 The index

The partition function for three black holes is, according to (5.38),

$$\mathcal{Z}^{(3)}(y,z) = \sum \mathrm{dim} H^2(\mathcal{M}_3, j_L, j_R) z^{2j_R} y^{2j_L} = 2\left[\mathcal{Z}^{(2)}(y,z)\right]^2 \tag{5.39}$$

and the superconformal index is

$$\mathcal{I}^{(3)}(y) = \sum \mathrm{dim} H^2(\mathcal{M}_3, j_L, j_R) (-)^{2j_R} y^{2j_L} = 2\left[\mathcal{I}^{(2)}(y)\right]^2\,. \tag{5.40}$$

6. Super-Poincare cohomology

In this paper we have defined the index $\mathcal{I}^{(N)}$ by exploiting the enhanced superconformal structure of low-energy black hole quantum mechanics. One may also define the index using only the superpoincare structure. This leads to the standard formula for the Witten index in supersymmetric quantum mechanics as the dimension of the kernel of $\partial + \partial^\dagger$ minus the dimension of the kernel of the adjoint. In order to make this well defined we must restrict to L_2 forms (without an e^{-2K} measure factor).

It is not hard to see, at least for $N = 2$, that there are no L_2 eigenstates of $(\partial + \partial^\dagger)^2$ and this definition leads to a trivial index, in contrast to the superconformal result (4.7). This comes about because the states which contribute to the index as computed in (4.7) in some sense live at the boundary of moduli space (where black holes coincide) and are lost in the restriction to L_2 states.

Potentially, these different definitions of the index are answers to different physical questions. Which definitions of the index will be useful for a full understanding of low-energy black hole dynamics remains to be seen.

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A. The behavior of L

In this appendix we detail some properties of L and verify the bound (3.7) used in the proof of the first vanishing theorem.

The expression (2.5) for the function L is divergent, but this divergence disappears after differentiating L to form the metric. The irrelevant divergences can be subtracted, yielding the finite (for $\vec{X}^A \neq \vec{X}^B$), but more complicated-looking, expression [6]

$$L = L_2 + L_3\,, \tag{A.1}$$

with

$$\begin{aligned} L_2 &= -6\pi^2 Q^3 \sum_{A\neq B}^{N} \frac{\ln|\vec{X}^A - \vec{X}^B|}{|\vec{X}^A - \vec{X}^B|^2}\,, \\ L_3 &= -Q^3 \sum_{A\neq B\neq C}^{N} \int d^4X \frac{1}{|\vec{X}-\vec{X}^A|^2|\vec{X}-\vec{X}^B|^2|\vec{X}-\vec{X}^C|^2}\,. \end{aligned} \tag{A.2}$$

These obey $\mathcal{L}_D L_3 = L_3$ and $\mathcal{L}_D L_2 = L_2 + 1/2K$.

We wish to show that the quantity

$$\frac{|\partial^- L|^2}{K \ln^2 K} \tag{A.3}$$

is bounded in the regions of small $K < 1/\mu$ and large $K > \mu$ for sufficiently large μ.

On a surface of constant K everything is bounded. Let us denote by Θ a set of coordinates on such a surface. The Lie derivative $\mathcal{L}_D$ generates motion to larger values of K as $\mathcal{L}_D = K\ d/dK$. First, note that $\mathcal{L}_D(L/K) = 1/2$, which may be integrated along orbits of $\mathcal{L}_D$ to give

$$\frac{L}{K} = \frac{1}{2}\ln K + f(\Theta)\,, \tag{A.4}$$

where f is some function of the angular coordinates. Now, using $I_- = -\partial\partial^- L$ we find that

$$\mathcal{L}_D \frac{|\partial^- L|^2}{K} = \frac{L}{K} + \frac{1}{2}\,. \tag{A.5}$$

Plugging in (A.4) we integrate to find $|\partial^- L|^2/K = 1/4 \ln^2 K + f(\Theta)\ln K + g(\Theta)$ where f and g are both functions only of Θ. We conclude that

$$\frac{|\partial^- L|^2}{K \ln^2 K} = \frac{1}{4} + \frac{f(\Theta)}{\ln K} + \frac{g(\Theta)}{\ln^2 K}\,. \tag{A.6}$$

This goes to $1/4$ at $K \to 0$ and at $K \to \infty$ since f and g are both bounded functions. Thus (A.3) is bounded at both small K and large K.

B. Ball cohomology

We now summarize various results on ball cohomology in four dimensions. As usual we break the cohomology groups into eigenspaces of J^3_L and $J^3_R - p/2$. Recall that we use ∂-cohomology rather than $\bar\partial$-cohomology.

Let B be the standard 4-ball. Then the only nonzero Neumann cohomology is

$$H^0(B, j_L, j_R) = \mathbf{C}\,, \qquad j_R \geq 0\,, \quad |j_L| \leq |j_R|\,, \quad \text{and } j_L + j_R \in \mathbb{Z}\,. \tag{B.1}$$

This may be seen by noting that anti-holomorphic functions on the 4-ball are generated by monomials of the form $(\bar z_1)^a (\bar z_2)^b$, where $j_R = (1/2)(a+b)$ and $j_L = (1/2)(a-b)$. Here (z_1, z_2) are the usual complex coordinates on R^4. Hodge duality exchanges Neumann and Dirichlet boundary conditions and takes $(j_L, j_R) \to (-j_L, -j_R)$. Thus the nonvanishing Dirichlet cohomology of B is

$$H^2_D(B, j_L, j_R) = \mathbf{C}\,, \qquad j_R \leq 0\,, \quad |j_L| \leq |j_R|\,, \qquad \text{and } j_L + j_R \in \mathbb{Z}\,. \tag{B.2}$$

Let B^* be the punctured ball, which is the standard 4-ball minus a smaller 4-ball centered at the origin. We use the short exact sequence relative to this decomposition

$$0 \longrightarrow \Omega_D(B, j_L, j_R) \longrightarrow \Omega(B, j_L, j_R) \longrightarrow \Omega(B^*, j_L, j_R) \longrightarrow 0 \tag{B.3}$$

which relates forms with Dirichlet and Neumann boundary conditions. As usual, this induces a long exact sequence giving the Neumann cohomology of B^* in terms of (B.1) and (B.2). The result is

$$\begin{aligned} H^0(B^*, j_L, j_R) &= \mathbf{C}\,, \qquad j_R \geq 0\,, \quad |j_L| \leq |j_R|\,, \qquad \text{and } j_L + j_R \in \mathbb{Z} \\ H^1(B^*, j_L, j_R) &= \mathbf{C}\,, \qquad j_R \leq -\frac{1}{2}\,, \quad |j_L| \leq |j_R|\,, \qquad \text{and } j_L + j_R \in \mathbb{Z}\,. \end{aligned} \tag{B.4}$$

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